Relationships Between Positioning Error Measures in Global Positioning System

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Introduction

IN Global Positioning System (GPS) positioning, various measures can be used to select satellites or to evaluate the positioning accuracy. Geometric dilution of precision (GDOP) and relative GDOP (RGDOP) are the most popular ones. Though these measures are used frequently, the relationship between them is not clear. The condition number is also a traditional measure of numerical stability in solving linear equations. All of these measures have some common properties and some differences. The relationships between these measures are analyzed in this Note after a brief survey of the GPS positioning problem.

Background

Absolute Positioning

The measured pseudorange to satellite i at receiver A is denoted by[1]

$$\Psi_A^i = r_A^i + cB_A + w_A^i \tag{1}$$

where a measured pseudorange Ψ_A^i is composed of a real range r_A^i, a distance proportional to the clock bias of the receiver cB_A, and the measurement error w_A^i. There are many sources of measurement errors, such as receiver noise, atmospheric delay, and intentional degradation of the signal called selective availability (SA). SA is the most dominant error source, and it typically causes a measurement error of 30 m (1σ). To reduce the effect of atmospheric delay, either the Klobuchar or the Hopfield model is usually adopted in GPS positioning.[1]

A linearized measurement equation at a nominal point is modeled as

$$\delta\Psi_A^i \equiv \Psi_A^i - r_{A0}^i = \boldsymbol{g}_{A0}^{i\,T}\delta\boldsymbol{x} + \delta(cB_A) + \nu_A^i \tag{2}$$

where r_{A0}^i is the calculated range, $g_{A0}^i - [g_{A0x}^i \;\; g_{A0y}^i \;\; g_{A0z}^i]^T$ is the line-of-sight vector, δx is the three-dimensional position error vector, and ν_A^i is the remaining error after compensation. In Eq. (2), there are four unknowns [$\delta\boldsymbol{x}$ and $\delta(cB_A)$], and measurements to at least four satellites are required to solve for all of them. Vectorizing Eq. (2) for n satellites, we have

$$\delta\boldsymbol{\Psi} = [G \quad \boldsymbol{r}]\begin{bmatrix}\delta\boldsymbol{x}\\ \delta(cB_A)\end{bmatrix} + \boldsymbol{\nu} = H\delta\boldsymbol{u} + \boldsymbol{\nu} \tag{3}$$

where

$$\delta\boldsymbol{\Psi} = \begin{bmatrix}\delta\Psi_A^1 & \cdots & \delta\Psi_A^n\end{bmatrix}^T, \qquad \boldsymbol{\nu} = \begin{bmatrix}\nu_A^1 & \cdots & \nu_A^n\end{bmatrix}^T$$

$$G = \begin{bmatrix}\boldsymbol{g}_{A0}^1 & \cdots & \boldsymbol{g}_{A0}^n\end{bmatrix}^T \in \Re^{n\times 3}, \qquad \boldsymbol{r} = [1 \quad \cdots \quad 1]^T$$

and

$$\delta\boldsymbol{u} = [\delta\boldsymbol{x} \quad \delta(cB_A)]^T$$

Relative Positioning

A position relative to a known reference point can be found with double-differenced measurements. The double-difference operation of the code delay measurements of two receivers R and A is defined as

$$\Psi_{RA}^{ij} \equiv \left(\Psi_A^j - \Psi_A^i\right) - \left(\Psi_R^j - \Psi_R^i\right) = r_{RA}^{ij} + w_{RA}^{ij} \tag{4}$$

where it is clear that the clock bias, atmospheric delay, and SA are effectively canceled. Double-difference operation, however, amplifies variances of errors such as multipath and receiver noise, which are relatively smaller than atmospheric delay and SA. It is well known that the magnitude of carrier-phase measurement error is smaller than that of pseudorange. Therefore, double-differenced carrier-phase measurements usually are used for relative positioning. The double-differenced relative positioning equations using carrier-phase measurements with resolved integer ambiguity are the same as for code measurements. A linearized measurement equation at nominal points R and A_0 is obtained as

$$\delta\Psi_{RA}^{ij} \equiv \Psi_{RA}^{ij} - r_{RA0}^{ij} = g_{A0}^{ij}\delta x + \varepsilon_{RA}^{ij} \tag{5}$$

where $g_{A0}^{ij} = g_{A0}^j - g_{A0}^i$, δx is the three-dimensional position error vector between A and A_0, and ε_{RA}^{ij} is the measurement error. For n satellites, the three-dimensional relative position can be obtained using the following linear equation:

$$\delta\boldsymbol{\Psi}_D = G_D\delta\boldsymbol{x} + \varepsilon_D \tag{6}$$

where

$$\delta\boldsymbol{\Psi}_D = \begin{bmatrix}\delta\Psi_{RA}^{12} & \cdots & \delta\Psi_{RA}^{(n-1)n}\end{bmatrix}^T$$

$$G_D = \begin{bmatrix}\boldsymbol{g}_{A0}^{12} & \cdots & \boldsymbol{g}_{A0}^{(n-1)n}\end{bmatrix}^T \in \Re^{(n-1)\times 3}$$

and

$$\boldsymbol{\varepsilon}_D = \begin{bmatrix}\varepsilon_{RA}^{12} & \cdots & \varepsilon_{RA}^{(n-1)n}\end{bmatrix}^T$$

Relationship Between GDOP and RGDOP

GDOP: Definition and Characteristics

Assume $\boldsymbol{\nu}$ in Eq. (3) is white Gaussian measurement noise with zero mean and covariance $Q = I_{n\times n}$. All components of $\boldsymbol{\nu}$ are assumed uncorrelated. Then, least-squares estimation gives

$$\delta\hat{\boldsymbol{u}} = (H^TQ^{-1}H)^{-1}H^TQ^{-1}\delta\boldsymbol{\Psi} \tag{7}$$

$$\mathrm{cov}(\delta\hat{\boldsymbol{u}}) = (H^TH)^{-1} \tag{8}$$

GDOP is defined as

$$\mathrm{GDOP} = \sqrt{\mathrm{tr}(H^TH)^{-1}} \tag{9}$$

where tr = trace of matrix.

GDOP is an indicator of estimation error in position and time per unit of measurement noise covariance and depends solely on the user-satellite geometry matrix H. For this reason, it often is used as the satellite selection criterion.[2] GDOP is independent of the coordinate system employed and can be represented as

$$\mathrm{GDOP} = \sqrt{\sigma_E^2 + \sigma_N^2 + \sigma_U^2 + \sigma_{CB}^2} \tag{10}$$

where σ_E^2, σ_N^2, σ_U^2, σ_{CB}^2 are the variances of the user position in three dimensions and the receiver clock bias. A positional DOP (PDOP)

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also can be defined for users who are not concerned about clock synchronization errors:

$$\text{PDOP} = \sqrt{\sigma_E^2 + \sigma_N^2 + \sigma_U^2} \tag{11}$$

The minimum value of GDOP is significant because the rms user error is the product of GDOP and measurement noise variance. Using an eigenvalue approach, Fang[3] showed that the minimum value of GDOP with four satellites is $\sqrt{2.5}$. This approach can be expanded easily to $n(>4)$ satellites. For a given H, define

$$\Xi \equiv H^T H = \begin{bmatrix} \sum_{i=1}^n g_{xi}^2 & \sum_{i=1}^n g_{xi} g_{yi} & \sum_{i=1}^n g_{xi} g_{zi} & \sum_{i=1}^n g_{xi} \\ \sum_{i=1}^n g_{xi} g_{yi} & \sum_{i=1}^n g_{yi}^2 & \sum_{i=1}^n g_{yi} g_{zi} & \sum_{i=1}^n g_{yi} \\ \sum_{i=1}^n g_{xi} g_{zi} & \sum_{i=1}^n g_{yi} g_{zi} & \sum_{i=1}^n g_{xi}^2 & \sum_{i=1}^n g_{zi} \\ \sum_{i=1}^n g_{xi} & \sum_{i=1}^n g_{yi} & \sum_{i=1}^n g_{zi} & n \end{bmatrix} \tag{12}$$

By the property of line-of-sight vector, we get $\text{tr}(\Xi) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 2n$, where λ_i are the eigenvalues of Ξ. Because Ξ is symmetric and positive definite, $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \lambda_4$ holds. Now, GDOP can be defined by eigenvalues of Ξ as

$$\text{GDOP}^2 = (1/\lambda_1) + (1/\lambda_2) + (1/\lambda_3) + (1/\lambda_4) \tag{13}$$

From the definition of maximum eigenvalue, we get $\lambda_4 \ge n$ and $\lambda_1 \le (n/3)$ (Ref. 3). These lead to an inequality relationship between GDOP and the eigenvalues of Ξ, as depicted in Eq. (14):

$$\text{GDOP} \ge \min\left\{\sqrt{\frac{1}{\lambda_4} + \frac{9}{2n - \lambda_4}}\right\} \tag{14}$$

The minimum of GDOP for n satellites occurs when $\lambda_n = n$ and is $(10/n)^{1/2}$. It is seen easily that, for the four-satellite case, the minimum of GDOP is $2.5^{1/2}$ with $\lambda_4 = 4$. Geometrically, four satellites that are located in the vertices of a regular tetrahedron give the minimum GDOP of $2.5^{1/2}$. The minimum value of GDOP, however, never occurs in reality because the Earth itself always masks some satellites. Considering the mask angle caused by the Earth itself, the minimum value of GDOP is $3^{1/2}$, which is larger than that of Fang's result. This occurs when three satellites are located on the horizon, forming a regular triangle, and the remaining satellite is located in the zenith.

RGDOP: Definition and Relation to GDOP

For n satellites, a linearized measurement equation at nominal points R and A_0 can be denoted as

$$D \cdot \Delta\boldsymbol{\Psi} = S \cdot G \cdot \delta\boldsymbol{x} + D \cdot \Delta\boldsymbol{\nu} \tag{15}$$

where

$$\Delta\boldsymbol{\Psi} = \begin{bmatrix} \Psi_R^1 & \Psi_A^1 & \Psi_R^2 & \Psi_A^2 & \cdots & \Psi_R^n & \Psi_A^n \end{bmatrix}$$

$$\Delta\boldsymbol{\nu} = \begin{bmatrix} \nu_R^1 & \nu_A^1 & \nu_R^2 & \nu_A^2 & \cdots & \nu_R^n & \nu_A^n \end{bmatrix}$$

G is a matrix defined in Eq. (3), and S and D are the single-difference and double-difference operators defined as Eqs. (16) and (17), respectively:

$$D = \begin{bmatrix} 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -1 & -1 & 1 \end{bmatrix} \in \Re^{(n-1)\times 2n} \tag{16}$$

$$S = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix} \in \Re^{(n-1)\times n} \tag{17}$$

User position can be found by applying least squares to Eq. (15), but it must be a weighted least squares because the measurement noise $D \cdot \Delta\boldsymbol{\nu}$ is correlated. Assuming $\Delta\boldsymbol{\nu}$ is white Gaussian with zero mean and covariance $Q = I_{2n\times 2n}$, the covariance of $D \cdot \Delta\boldsymbol{\nu}$ is $Q_D \equiv DD^T \in \Re^{(n-1)\times(n-1)}$. Using the covariance of an estimated position $\delta\hat{\boldsymbol{x}}$, RGDOP is defined as

$$\text{RGDOP} \equiv \sqrt{\text{tr}[\text{cov}(\delta\hat{\boldsymbol{x}})]} = \sqrt{\text{tr}\{[G^T S^T (DD^T)^{-1} SG]^{-1}\}} \tag{18}$$

RGDOP is an indicator of relative positioning error, whereas GDOP is that of absolute positioning.

By premultiplying Eq. (3) by S, we get

$$S\delta\boldsymbol{\Psi} = S[G \quad \boldsymbol{r}]\begin{bmatrix} \delta\boldsymbol{x} \\ cB_A \end{bmatrix} + S\boldsymbol{\nu} = SG\delta\boldsymbol{x} = S\boldsymbol{\nu} \tag{19}$$

where clock bias is removed by single-difference operation. Under the same assumption of measurement noise, the covariance of $S\boldsymbol{\nu}$ is $Q_S = SS^T \in \Re^{(n-1)\times(n-1)}$. Using the estimates of position $\delta\hat{\boldsymbol{x}}$, PDOP can be defined as

$$\text{PDOP} \equiv \sqrt{\text{tr}[\text{cov}(\delta\hat{\boldsymbol{x}})]} = \sqrt{\text{tr}\{[G^T S^T (SS^T)^{-1} SG]^{-1}\}} \tag{20}$$

The equality $DD^T = 2SS^T$ gives the following relationship between PDOP and RGDOP:

$$\text{RGDOP} = \sqrt{2} \times \text{PDOP} \tag{21}$$

This shows that RGDOP and PDOP are proportional. Thus, PDOP, obtained in absolute positioning, can be used instead of RGDOP. The minimum value of RGDOP is $(18/n)^{1/2}$ when GDOP is $(10/n)^{1/2}$. Considering the mask angle caused by the Earth itself, the minimum value of RGDOP is $2(4/3)^{1/2}$ for four satellites.

Relationship Between GDOP and Condition Number

The condition number of a square matrix is a measure of the numerical stability of the matrix inverse used in solving linear equations.[4] For the positive-definite matrix Ξ, condition number is defined as $\kappa(\Xi)$, where

$$\kappa(\Xi) = \lambda_4/\lambda_1 \tag{22}$$

From Eqs. (13) and (22) and the fact that $n \le \lambda_4 < 2n$, the following relationships are derived:

$$(1/\lambda_4)[\kappa(\Xi) + 3] \le \text{GDOP}(\Xi)^2 \le (1/\lambda_4)[3\kappa(\Xi) + 1] \tag{23}$$

$$\frac{n\text{GDOP}(\Xi)^2 - 1}{3} \le \kappa(\Xi) < 2n\text{GDOP}(\Xi)^2 - 3 \tag{24}$$

Numerical Example

From Eq. (24), for the four-satellite case, the lower bound of the condition number is 3 because the minimum of GDOP is $2.5^{1/2}$. A minimum GDOP is obtained with the following matrix:

$$H = \begin{bmatrix} 0.6005 & -0.5542 & -0.5764 & 1 \\ -0.5542 & 0.6005 & -0.5764 & 1 \\ -0.5996 & -0.5996 & 0.5301 & 1 \\ 0.5533 & 0.5533 & 0.6227 & 1 \end{bmatrix} \tag{25}$$

The eigenvalues, condition number, and RGDOP are obtained as

$$\lambda(\Xi) = \{1.3333, 1.3333, 1.3333, 4\} \tag{26}$$

$$\kappa(\Xi) = \lambda_4/\lambda_1 = 3 \tag{27}$$

$$\text{RGDOP} = \sqrt{2} \times \text{PDOP} = \sqrt{2} \times \sqrt{\tfrac{9}{4}} \tag{28}$$

This example shows that a minimum of the condition number and RGDOP also can be found when four satellites are located in the vertices of a regular tetrahedron.

Conclusion

In this Note, three measures that can be used in GPS positioning are introduced and detailed. Inequalities describing the relationship between GDOP and condition number are derived, as well as the relationship between GDOP and RGDOP. The results show that GDOP is approximately proportional to the condition number, whereas PDOP is exactly proportional to RGDOP. These results will enhance the understanding of the mathematical aspect of GPS positioning and can be applied to satellite selection and constellation design problems.

References

[1]Hofmann-Wellenhof, B., Lichtenegger, H., and Collins, J., *Global Positioning System: Theory and Practice*, Springer–Verlag, Vienna, 1993, pp. 99–117.

[2]Siouris, G. M., *Aerospace Avionics Systems—A Modern Synthesis*, Academic, San Diego, CA, 1993, p. 311.

[3]Fang, B. T., "The Minimum for Geometric Dilution of Precision in Global Positioning System Navigation," *Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 1, 1987, p. 116.

[4]Golub, G. H., and Van Loan, C. F., *Matrix Computations*, Johns Hopkins Univ. Press, Baltimore, MD, 1987, pp. 79–81.

Extension of the Friedland Parameter Estimator to Discrete-Time Systems

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I. Introduction

FRIEDLAND[1] developed a globally convergent nonlinear observer to estimate parameters for continuous-time systems that are affine in the unknown parameter. The derivation of the Friedland observer requires smooth partial derivatives and is, therefore, inherently limited to continuous-time systems. Inasmuch as the realization of the observer will use discrete-time digital sampling, it is desirable to develop a direct discrete-time implementation.

In this Engineering Note, the discrete-time version of Friedland's parameter estimator is derived and extended to the case of time-varying parameters. Estimation of the angular rate of a momentum wheel from quadrature resolver position output is demonstrated. Estimation of the poles of an autoregressive filter is also demonstrated, and the conventional solution is shown to be a subset of this more general technique.

II. Derivation

Consider the discrete-time dynamic system

$$\boldsymbol{x}_{k+1} = f(\boldsymbol{x}_k, \boldsymbol{u}_k, \boldsymbol{p}_k) \tag{1}$$

$$\boldsymbol{p}_{k+1} = A(\boldsymbol{x}_k)\boldsymbol{p}_k \tag{2}$$

where $\boldsymbol{x}$ is the state of the process, $\boldsymbol{u}$ is the control input, $f()$ is the state transition function, $\boldsymbol{p}$ is a vector of parameters, and $A(\boldsymbol{x}_k)$ is the time-dependent (or possibly state-dependent) parameter dynamics matrix. The entire state vector $\boldsymbol{x}_k$ is presumed available for direct measurement.

If the parameter-dependent part of the dynamics are affine in the parameter vector $\boldsymbol{p}$, Eq. (1) can be written

$$\boldsymbol{x}_{k+1} = F(\boldsymbol{x}_k, \boldsymbol{u}_k)\boldsymbol{p}_k + g(\boldsymbol{x}_k, \boldsymbol{u}_k) \tag{3}$$

$$\boldsymbol{p}_{k+1} = A(\boldsymbol{x}_k)\boldsymbol{p}_k \tag{4}$$

The proposed observer for parameter vector $\boldsymbol{p}$ has the form of a reduced-order estimator:

$$\hat{\boldsymbol{p}}_k = A(\boldsymbol{x}_{k-1})\hat{\boldsymbol{p}}_{k-1} + M(\boldsymbol{x}_{k-1})\boldsymbol{x}_k + z_k \tag{5}$$

$$z_{k+1} = -M(\boldsymbol{x}_k)[F(\boldsymbol{x}_k, \boldsymbol{u}_k)\hat{\boldsymbol{p}}_k + g(\boldsymbol{x}_k, \boldsymbol{u}_k)] \tag{6}$$

where $\hat{\boldsymbol{p}}$ is the estimate of $\boldsymbol{p}$, $M(\boldsymbol{x})$ is a state-dependent gain matrix, and z is an intermediate state variable of the same dimension as $\boldsymbol{p}$.

The dynamics of the parameter estimation error $\epsilon_{k+1} = \hat{\boldsymbol{p}}_{k+1} - \boldsymbol{p}_{k+1}$ can be expressed as

$$\epsilon_{k+1} = A(\boldsymbol{x}_k)\hat{\boldsymbol{p}}_k + M(\boldsymbol{x}_k)\boldsymbol{x}_{k+1} + z_{k+1} - A_k\boldsymbol{p}_k \tag{7}$$

$$\begin{aligned}\epsilon_{k+1} = A(\boldsymbol{x}_k)\epsilon_k + M(\boldsymbol{x}_k)[F(\boldsymbol{x}_k, \boldsymbol{u}_k)\boldsymbol{p}_k + g(\boldsymbol{x}_k, \boldsymbol{u}_k)] \\ - M(\boldsymbol{x}_k)[F(\boldsymbol{x}_k, \boldsymbol{u}_k)\hat{\boldsymbol{p}}_k + g(\boldsymbol{x}, \boldsymbol{u})]\end{aligned} \tag{8}$$

$$\epsilon_{k+1} = [A(\boldsymbol{x}_k) - M(\boldsymbol{x}_k)F(\boldsymbol{x}_k, \boldsymbol{u}_k)]\epsilon_k \tag{9}$$

which is stable when the eigenvalues of matrix

$$A(\boldsymbol{x}_k) - M(\boldsymbol{x}_k)F(\boldsymbol{x}_k, \boldsymbol{u}_k)$$

are inside the unit circle. The observer design problem is to find a matrix $M(\boldsymbol{x}_k)$ that produces this result.

III. Unmodeled Parameter Dynamics

The effect of an error in the model for the parameter dynamics [Eq. (4)] is to cause a steady-state estimations error. The true plant is given by Eqs. (3) and (4). The observer is constructed as in Eqs. (5) and (6) but with the true $A(\boldsymbol{x}_{k-1})$ replaced by an erroneous $\tilde{A}(\boldsymbol{x}_{k-1})$:

$$\hat{\boldsymbol{p}}_k = \tilde{A}(\boldsymbol{x}_{k-1})\hat{\boldsymbol{p}}_{k-1} + M(\boldsymbol{x}_{k-1})\boldsymbol{x}_k + z_k \tag{10}$$

The resulting error dynamics are given as

$$\epsilon_{k+1} = [\tilde{A}(\boldsymbol{x}_k) - M(\boldsymbol{x}_k)F(\boldsymbol{x}_k, \boldsymbol{u}_k)]\epsilon_k + [\tilde{A}(\boldsymbol{x}_k) - A(\boldsymbol{x}_k)]\boldsymbol{p}_k \tag{11}$$

In steady state, $\epsilon_{k+1} = \epsilon_k$. From Eq. (11), the steady-state error ϵ_∞ is given as

$$\epsilon_\infty = [I - \tilde{A}(\boldsymbol{x}_\infty) + M(\boldsymbol{x}_\infty)F(\boldsymbol{x}_\infty, \boldsymbol{u}_\infty)]^{-1}[\tilde{A}(\boldsymbol{x}_\infty) - A(\boldsymbol{x}_\infty)]\boldsymbol{p} \tag{12}$$

Equation (12) implies that, if the parameter is bounded, then so is the estimation error if $[\tilde{A}(\boldsymbol{x}_k) - M(\boldsymbol{x}_k)F(\boldsymbol{x}_k, \boldsymbol{u}_k)]$ is a stable matrix.

IV. Example 1: Spacecraft Momentum Wheel Rate Estimator

The sine and cosine of angular position of a spacecraft momentum wheel is measured using a resolver with quadrature output. The state is directly measured and is a function of time and wheel velocity:

$$\boldsymbol{x}_k = \begin{bmatrix}\cos\omega kT \\ \sin\omega kT\end{bmatrix} \tag{13}$$

where ω is the wheel angular rate, k is the discrete-time index, and $T = 1$ s is the sample period.

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